Moschovakis Coding lemma

Reference :
[1] Steve Jackson : Structural consequences of AD

+V

thm (AD+DC$_\mathbb{R}$)([1])

Let $\utilde{\Gamma}$ be a non-self dual pointclass closed under $\exists^\mathbb{R}$ and $\wedge$

Let $\prec$ be a $\utilde{\Gamma}$-pwo relation on $\mathbb{R}$ of rank $\theta \in ON$

Let $R \subseteq dom(\prec) \times \mathbb{R}$ be <u>any</u> relation s.t.

$\forall x \in dom(\prec) \exists y\ R(x,y)$.

Then there is a $\utilde{\Gamma}$ set $A \subseteq dom(\prec) \times \mathbb{R}$ s.t.

(1) $\forall \alpha < \theta\ \exists x \in dom(\prec)\ \exists y\ (|x|_\prec = \alpha \wedge A(x,y))$

(2) $\forall x\ \forall y\ (A(x,y) \to (x \in dom(\prec) \wedge R(x,y)))$

(Hence A is a <u>definable</u> choice set of R)

(proof)

We need the following fact :

Fact there are Γ-universal sets $U_n \subseteq \mathbb{R} \times \mathbb{R}^n$ in Γ for each $n \in \omega$, with the following properties :

(1) (s-m-n theorem)

for $m > n$, there is a continuous function $S_{m,n} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ s.t.

$U_m(y, x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) \Leftrightarrow U_{m-n}(S_{m,n}(y, x_1, \cdots, x_n), x_{n+1}, \cdots, x_m)$

(2) (Recursion theorem)

for every $n \in \omega$, and for every Γ-set $A \subseteq \mathbb{R} \times \mathbb{R}^n$, there is $y^* \in \mathbb{R}$ s.t. for all $y \in \mathbb{R}^n$.

$$U_n(y^*, y) \iff A(y^*, y)$$

Sketch

Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be any Γ-universal set in Γ.

Fix a recursive bijection $y \mapsto (y_0, y_1)$ between $\mathbb{R}$ and $\mathbb{R}^2$.

Define $U_n(y, (x_1, \cdots, x_n)) :\iff U(y_0, \langle y_1, x_1, \cdots, x_n \rangle)$.

$(U_n \mid n \in \omega)$ work.

Fix $\{U_n \mid n \in \omega\}$ as above.

Suppose that the theorem fails.

Let θ be the minimal witness for that w/ $<, R$.

Then θ is a limit ordinal.

For $\delta < \theta$, we say $x \in \mathbb{R}$ codes a δ-choice set if $U_2(x, -, -)$ satisfies ① w/ for all $\alpha \leq \delta$ and ②.

By assumption, for all $\delta < \theta$, there is $x \in \mathbb{R}$ which codes a δ-choice set.

Consider the following game : let I, II play x, y.

I	x
II	y

II wins iff (if x codes a δ-choice set for some $\delta < \Theta$, then y codes a δ'-choice set for some $\delta' \in [\delta, \Theta)$)

(Hence, this is a Solovay type game)

Case 1 player I wins :

let σ be a winning strategy for I.

Let $B = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} \ x = \sigma[y] \} \in \underset{\sim}{\Sigma}^1_1$.

Then for any $\delta < \Theta$, there is $x \in B$ s.t.

x codes a δ'-choice set w/ $\delta' \in [\delta, \Theta)$

So define

$$A(x, y) \Longleftrightarrow \exists z \in B \ U_2(z, x, y).$$

Obviously A works. //

Case 2 player II wins :

let τ be a winning strategy for II.

Observation

(i) If we input a code x of a δ-choice set, then the output $y = \tau[x]$ should code a larger choice set.

(ii) Hence we aim to "recursively" construct $\mho_2(\varepsilon_0, -, -)$ s.t. $\mho_2(\varepsilon_0, x, z) \Leftrightarrow$ "z codes a $\geq |x|_{\prec}$-choice set."

(iii) In that case, z should be of the form $z = \tau[(\text{something})]$. We aim to find an appropriate (something).

Fix some $z_0 \in \mathbb{R}$ that codes a δ-choice set some $0 < \delta < \theta$.

Define

$$C(\varepsilon, x, t, w) :\Leftrightarrow \mho_2(z_0, t, w) \vee \exists y \exists z \left(y \prec x \wedge \mho_2(\varepsilon, y, z) \wedge \mho_2(z, t, w) \right) \in \underset{\sim}{\Gamma}$$

Let $s : \mathbb{R}^2 \to \mathbb{R}$ be continuous s.t.

$$\mho_2(s(\varepsilon, x), t, w) \Leftrightarrow C(\varepsilon, x, t, w).$$

By the recursion theorem, let ε_0 be s.t.

$$\mho_2(\varepsilon_0, x, z) \Leftrightarrow \text{"} z = \tau[s(\varepsilon_0, x)] \text{"}$$

Claim

① $\forall x \in \mathrm{dom}(\prec)\ \exists! z\ \mho_2(\varepsilon_0, x, z)$

② $\forall x \in \mathrm{dom}(\prec)\ \forall z$

$$\left(\mho_2(\varepsilon_0, x, z) \Rightarrow z \text{ codes a } \geq |x|_\prec\text{-choice set}\right)$$

∵ ① clear

② By induction on $|x|_\prec$:

base step : clear, since we put $\mho_2(\varepsilon_0, t, w)$ in the def of $C(\varepsilon_0, x, t, w)$

the rest is eacy induction.

Now define

$$A(x, y) \Longleftrightarrow \exists z \in \mathrm{dom}(\prec)\ \exists w\ \left(\mho(\varepsilon_0, z, w) \wedge \mho(w, x, y)\right)$$

$$\in \underset{\sim}{\Gamma}$$

Clearly, A works. // ⊣ thm ■

Note we used the fact that $\underset{\sim}{\Gamma}$ is closed under $\forall$.

But thm 2.12 in ~~the~~ Jackson's HST article doesn't require that. Can we drop the assumption that $\underset{\sim}{\Gamma}$ is closed under $\forall$?

§ Application of Moschovakis Coding lemma

One typical application in DST is the following :

lemma

> Assume $AD + DC_{\mathbb{R}}$.
>
> Let $\boldsymbol{\Gamma}$ be non-self dual and closed under $\forall^{\mathbb{R}}, \wedge, \vee$, and assume $\mathrm{pwo}(\boldsymbol{\Gamma})$.
>
> Then any $\check{\boldsymbol{\Gamma}}$-well-founded relation has length less than $\delta(\boldsymbol{\Gamma})$.
>
> ($\delta(\boldsymbol{\Gamma}) :=$ the sup of the lengths of the $\boldsymbol{\Delta}$-pwo)

(proof)

Let P be a $\boldsymbol{\Gamma}$-cplt set.

Let φ be a regular $\boldsymbol{\Gamma}$-norm on P.

Now suppose that $\prec$ is a $\check{\boldsymbol{\Gamma}}$-well-founded relation of length $\geq |\varphi|$. WMAT $|\prec| = |\varphi|$.

Define $R \subseteq \mathrm{dom}(\prec) \times \mathbb{R}$ by

$$R(x, y) \Leftrightarrow (y \in P \wedge \varphi(y) = |x|_{\prec})$$

By Coding lemma (You may prove Moschovakis Coding lemma when $\prec$ is a well-founded relation.),

let $A \subseteq \mathrm{dom}(\prec) \times \mathbb{R}$ be a choice set in $\check{\boldsymbol{\Gamma}}$.

Let $B(y) \iff \exists x \; A(x,y) \quad \in \check{\boldsymbol{\Gamma}}$.

Then $P(x) \iff \exists y \left(B(y) \wedge x \prec_{\check{\boldsymbol{\Gamma}}} y \right) \in \check{\boldsymbol{\Gamma}}$

This is a contradiction !!

Recall) $\Theta = \sup\{ \alpha \in ON \mid \exists f : \mathbb{R} \twoheadrightarrow \alpha \text{ surjective} \}$

thm (This is rather an application of the proof of coding lemma than an application of coding lemma.)

Assume $AD + DC_{\mathbb{R}}$.

Let $\lambda < \Theta$.

Then

there is a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\lambda)$.

In particular, Θ is a limit cardinal.

(proof) Fix a surjection $f : \mathbb{R} \twoheadrightarrow \lambda$.

We aim to define a sequence $\langle g_\xi \mid \xi \leq \lambda \rangle$ recursively s.t. each $g_\xi : \mathbb{R} \to \mathcal{P}(\xi)$ is a surjection.

g_0 : clear

$g_\xi \mapsto g_{\xi+1}$: use $\xi \xrightarrow{\text{a bijection}} \xi+1$, defined uniformly on ξ.

Now suppose δ is a limit ordinal. Fix $X \subseteq \delta$.

Consider the following game $G(X)$:

let I play x_0, x_1

II play y_0, y_1

~~II wins iff~~ $\cancel{\left(\text{if } g_{f(x_0)}(x_1) = X \cap f(x_0), \text{ then } f(y_0) > f(x_0) \text{ and } g_{f(y_0)}(y_1) = X \cap f(y_0)\right)}$

~~(again, this is a Solovay type game)~~

II wins iff $\left(\begin{array}{l} \text{if } f(x_0) < \delta \text{ and } g_{f(x_0)}(x_1) = X \cap f(x_0), \\ \text{then } f(x_0) < f(y_0) < \delta \text{ and} \\ \qquad g_{f(y_0)}(y_1) = X \cap f(y_0). \end{array}\right)$

(again, this is a Solovay Type game)

It suffices to show the following: let.

$g_\delta : \mathbb{R} \to P(\delta)$:

$$g_\delta(x) = \begin{cases} X & \text{if } x \text{ is a winning strategy for } G(X) \\ \emptyset & \text{o.w.} \end{cases},$$

then g_δ is a well-defined surjection.

Now suppose that x is a winning strategy for both $G(X)$ and $G(Y)$.

case1 x is a winning strategy for I in the game $G(X)$:

if $X \neq Y$, then let $\xi \in X \triangle Y$.

Let y_0 be s.t. $f(y_0) = \xi + 1$,

and let y_1 be s.t. $g_{f(y_0)}(y_1) = X \cap (\xi + 1)$.

Then $x * \langle y_0, y_1 \rangle$ must "code" $X \cap \mathcal{Y}$ for some $\mathcal{Y} \geq \xi$,

namely let $x_0 = (x * \langle y_0, y_1 \rangle)_0$, $x_1 = (x * \langle y_0, y_1 \rangle)_1$,

then $f(x_0) \in [f(y_0), \delta)$ and

$$g_{f(x_0)}(x_1) = X \cap f(x_0).$$

This implies x cannot be a winning strategy for $G(Y)$. A contradiction!!

Case 2 $\mathcal{X}$ is a winning strategy for II in the game $G(X)$:

if $X \neq Y$, then let $\xi \in X \triangle Y$.

let x_0 be s.t. $f(x_0) = \xi + 1$,

and let x_1 be s.t. $g_{f(x_0)}(x_1) = X \cap (\xi + 1)$.

Then let
$$y_0 = (\mathcal{X} * (x_0, x_1))_0$$
$$y_1 = (\mathcal{X} * (x_0, x_1))_1.$$

We have $f(y_0) \in (f(x_0), \delta)$
$$g_{f(y_0)}(y_1) = X \cap f(y_0).$$

Again, this implies $\mathcal{X}$ cannot be a winning strategy for $G(Y)$. A contradiction !!

Hence we have $X = Y$.

Therefore g_δ is well-defined, and is clearly surjective. $\dashv$ thm

∎

thm

Let $M \subseteq N$ be models of $ZF + AD + DC_{\mathbb{R}}$ s.t. $\mathbb{R}^M = \mathbb{R}^N$.

Let $\lambda < \Theta^M$.

Then $\mathcal{P}(\lambda) \cap M = \mathcal{P}(\lambda) \cap N$

(M, N have the same powersets below Θ^M.)

(proof) By induction on $\lambda < \Theta^M$. Note that $\Theta^M \leq \Theta^N$, since $\mathbb{R}^M = \mathbb{R}^N$.

Suppose that for all $\xi < \lambda$,

$$\mathcal{P}(\xi) \cap M = \mathcal{P}(\xi) \cap N.$$

WMAT λ is a limit ordinal. Fix a surjection $f : \mathbb{R} \twoheadrightarrow \lambda \in M$

Then by the previous theorem, we have that

$$\langle g^M_\xi \mid \xi < \lambda \rangle = \langle g^N_\xi \mid \xi < \lambda \rangle.$$

Here we use f and $\langle g^M_\xi \mid \xi < \lambda \rangle = \langle g^N_\xi \mid \xi < \lambda \rangle$.

Now fix any $X \in \mathcal{P}(\lambda) \cap N$.

Let $x \in \mathbb{R}^N$ be a winning strategy for $G(X)$.

Since $\mathbb{R}^M = \mathbb{R}^N$, $x \in M$.

Then X is computed from x, f, and $\langle g^M_\xi \mid \xi < \lambda \rangle$ inside of M exactly the same as we did in the proof of the previous theorem. Hence $X \in M$. // ⊣ thm $\blacksquare$